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Probabilities, Intervals, What Next? Optimization Problems Related to Extension of Interval Computations to Situations with Partial Information about Probabilities

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Abstract. When we have only interval ranges $[\underline{x}_i, \overline{x}_i]$ of sample values $x_1, \ldots, x_n$, what is the interval $[\underline{V}, \overline{V}]$ of possible values for the variance V of these values? We show that the problem of computing the upper bound $\overline{V}$ is NP-hard. We provide a feasible (quadratic time) algorithm for computing the exact lower bound $\underline{V}$ on the variance of interval data. We also provide feasible algorithms that computes $\overline{V}$ under reasonable easily verifiable conditions, in particular, in case interval uncertainty is introduced to maintain privacy in a statistical database. We also extend the main formulas of interval arithmetic for different arithmetic operations x_1 op x_2 to the case when, for each input x_i, in addition to the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ of possible values, we also know its mean E_i (or an interval $\mathbf{E}_i$ of possible values of the mean), and we want to find the corresponding bounds for $y = x_1$ op x_2 and its mean. In this case, we are interested not only in the bounds for y, but also in the bounds for the mean of y. We formulate and solve the corresponding optimization problems, and describe remaining open problems.

Key words: Interval computations, Robust statistics, Optimization

1. Introduction: data processing — from computing to probabilities to interval

1.1. WHY DATA PROCESSING?

In many real-life situations, we are interested in the value of a physical quantity y that is difficult or impossible to measure directly. Examples of such quantities are the distance to a star and the amount of oil in a given well. Since we cannot measure y directly, a natural idea is to measure y *indirectly*. Specifically, we find some easier-to-measure quantities $x_1, \ldots, x_n$ which are related to y by a known relation $y = f(x_1, \ldots, x_n)$; this relation may be a simple functional transformation, or complex algorithm (e.g., for the amount of oil, numerical solution to an inverse problem). Then, to estimate y, we first measure the values of the quantities

$x_i, \ldots, x_n$, and then we use the results $\tilde{x}_1, \ldots, \tilde{x}_n$ of these measurements to to compute and estimate $\tilde{y}$ for y as $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$.

For example, to find the resistance R, we measure current I and voltage V, and then use the known relation $R = V/I$ to estimate resistance as $\tilde{R} = \tilde{V}/\tilde{I}$.

Computing as estimate for y based on the results of direct measurements is called *data processing*; data processing is the main reason why computers were invented in the first place, and data processing is still one of the main uses of computers as number crunching divices

Comment. In this paper, for simplicity, we consider the case when the relation between x_i and y is known exactly; in some practical situations, we only know an approximate relation between x_i and y

1.2. WHY INTERVAL COMPUTATIONS? FROM COMPUTING TO PROBABILITIES TO INTERVALS

Measurement are never 100% accurate, so in reality, the value x_i of ith measured quantity can differ form the measurement result $\tilde{x}_i$. Because of these *measurement errors* $\Delta x_i \stackrel{\text{def}}{=} \tilde{x}_i - x_i$, the result $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$ of data processing is, in general, different from the actual value $y = (x_1, \ldots, x_n)$ of the desired quantity y [14].

It is desirable to describe the error $\Delta y \stackrel{\text{def}}{=} \tilde{y} - y$ of the result of data processing. To do that, we must have some information about the errors of direct measurement.

What do we know about the errors Δx_i of direct measurements? First, the manufacturer of the measuring instrument must supply us with an upper bound Δ_i on the measurement error. If no such upper bound is supplied, this means that no accuracy is guaranteed, and the corresponding "measuring instrument" is practically useless. In this case, once we performed a measurement and got a measurement result $\tilde{x}_i$, we know that the actual (unknown) value x_i of the measured quantity belongs to the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$, where $\underline{x}_i = \tilde{x}_i - \Delta_i$ and $\overline{x}_i = \tilde{x}_i + \Delta_i$.

In many practical situations, we not only know the interval $[-\Delta_i, \Delta_i]$ of possible values of the measurement error; we also know the probability of different values Δx_i within this interval. The knowledge underlies the traditional engineering approach to estimating the error of indirect measurement, in which we assume that we know the probability distributions for measurement errors Δx_i.

In practice, we can determine the desired probabilities of different values of Δx_i by comparing the results of measuring with this instrument with the results of measuring the same quantity by a standard (much more accurate) measuring instrument. Since the standard measuring instrument is much more accurate than the one used, the difference between these two measurement results is practically equal to the measurement error; thus, the empirical distribution of this difference is close to the desired probability distribution for measurement error. The are two cases, however, when this determination is not done:

- First is the case of cutting-edge measurements, e.g., measurements in fundamental science. When a Hubble telescope detects the light from a distant galaxy, there is no "standard" (much more accurate) telescope floating nearby that we can use to calibrate the Hubble: the Hubble telescope is the best we have.
- The second case is the case of measurements on the shop floor. In this case, in principle, every sensor can be thoroughly calibrated, but sensor calibration is so costly — usually costing ten times more than the sensor itself — that manufacturers rarely do it.

In both cases, we have no information about the probabilities of Δx_i; the only information we have is the upper bound on the measurement error.

In this case, after we performed a measurement and got a measurement result $\tilde{x}_i$, the only information that we gave about the actual value x_i of the measured quantity is that it belongs to the interval $\mathbf{x}_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$. In such situations, the only information that we have about the (unknown) actual value of $y = f(x_1, \ldots, x_n)$ is that y belongs to the range $\mathbf{y} = [\underline{y}, \overline{y}]$ of the function f over the box $\mathbf{x}_1 \times \cdots \times \mathbf{x}_n$:

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) \mid x_1 \in \mathbf{x}_1, \ldots, x_n \in \mathbf{x}_n\}$$

The process of computing this interval range based on the input intervals $\mathbf{x}_i$ is called *interval computations*; see e.g. [3–5,10].

1.3. INTERVAL COMPUTATIONS AS AN OPTIMIZATION PROBLEM

The main problem of interval computations can be naturally reformulated as an optimization problem. Indeed, $\underline{y}$ is the solution to the following problem: $f(x_1, \ldots, x_n) \to \min$, under the conditions

$$\underline{x}_1 \leqslant x_1 \leqslant \overline{x}_1; \cdots \underline{x}_n \leqslant x_n \leqslant \overline{x}_n,$$

and $\overline{y}$ is the condition to the maximization problem $f(x_1, \ldots, x_n) \to \max$ under the same conditions.

1.4. INTERVAL COMPUTATIONS TECHNIQUES: BRIEF REMINDER

Historically, the first method for computing the enclosure for the range is the method which is sometimes called "straightforward" interval computations. This method is based on the fact that inside the computer, every algorithm consists of elementary operations (arithmetic operations, min, max, etc.). For each elementary operation $f(a, b)$, if we know the intervals $\mathbf{a}$ and $\mathbf{b}$ for a and b, we can compute the exact range $f(\mathbf{a}, \mathbf{b})$. The corresponding formulas form the so-called *interval arithmetic*. For example

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]; \quad [\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}];$$
$$[\underline{a}, \overline{a}] \cdot [\underline{b}, \overline{b}] = [\min(\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}), \max(\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b})]$$

In straightforward interval computations, we repeat the computations forming the program f step-by-step, replacing each operation with real numbers by the corresponding operation of interval arithmetic. It is known that, as a result, we get an enclosure $\mathbf{Y} \supseteq \mathbf{y}$ for the desired range.

In some cases, this enclosure is exact. In more complex cases (see examples below), the enclosure has excess width.

There exist more sophisticated techniques for producing a narrower enclosure, e.g., a centered form method. However, for each of these techniques, there are cases when we get an excess width. Reason: as shown in [7,16], the problem of computing the exact range is known to be NP-hard even for polynomial functions $f(x_1, \cdots, x_n)$ (actually, even for quadratic functions f).

1.5. WHAT WE ARE PLANNING TO DO?

First, we analyze a specific interval computations problem — when we use traditional statistical data processing algorithms $f(x_1, \cdots, x_n)$ to process the results of direct measurements.

Then, we extend our analysis to the case when for each input x_i, in addition to the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ of possible values, we have partial information about the probabilities: specifically, we know its mean E_i (or an interval $\mathbf{E}_i$ of possible values of the mean).

We formulate and solve the corresponding optimization problems, and describe remaining open problems.

2. First step beyond intervals: error estimation for traditional statistical data processing algorithms under interval uncertainty

When we have n results $x_1, \ldots, x_n$ of repeated measurement of the same quantity (at different points, or at different moments on time), traditional statistical approach usually starts with computing their sample average $E = (x_1 + \cdots + x_n)/n$ and their (sample) variance

$$V = \frac{(x_1 - E)^2 + \cdots + (x_n - E)^2}{n} \tag{1}$$

(or, equivalently, the sample standard deviation $\sigma = \sqrt{V}$); see e.g. [14].

In this section, we consider situations when we do not know the exact values of the quantities $x_1, \ldots, x_n$, we only know the intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of possible values of x_i. In such situations, for different possible values $x_i \in \mathbf{x}_i$, we get different values of E and V. The question is: what are the intervals $\mathbf{E}$ and $\mathbf{E}$ of possible values of E and V.

The practical importance of this question was emphasized, e.g. in [11,12] on the example of processing geophysical data.

For E, the straightforward interval computations lead to the exact range:

$$\mathbf{E}=\frac{\mathbf{x}_1+\cdots+\mathbf{x}_n}{n},\quad \text{i.e., } \underline{E}=\frac{\underline{x}_1+\cdots+\underline{x}_n}{n},\quad \text{and}\quad \overline{E}=\frac{\overline{x}_1+\cdots+\overline{x}_n}{n}.$$

For V, straightforward interval computations lead to an excess width. For example, for $\mathbf{x}_1=\mathbf{x}_2=[0,1]$, the variance is $V=(x_1-x_2)^2/4$ and hence, the actual range $\mathbf{V}=[0,0.25]$. On the other hand, $\mathbf{E}=[0,1]$, hence

$$\frac{(\mathbf{x}_1-\mathbf{E}^2+(\mathbf{x}_2-\mathbf{E})^2}{2}=[0,1]\supset[0,0.25].$$

More sophisticated methods of interval computations also sometimes lead to an excess width, and the reason for this as that the corresponding optimization problem is NP-hard:

THEOREM 1. *Computing $\overline{V}$ is NP-hard.*

Comment. The main ideas of the proofs of the results from this section are given in [1].

The very fact that computing the range of a quadratic function is NP-hard was first proven by Vavasis [16] (see also [7]). We have shown that this difficulty happens even for very simple quadratic functions frequently used in data processing.

A natural question is: maybe the difficulty comes from the requirement that the range be computed exactly? In practice, it is often sufficient to compute, in a reasonable amount of time, a usefully accurate estimate $\tilde{\overline{V}}$ for $\overline{V}$, i.e., and estimate $\tilde{\overline{V}}$ which is accurate with a giver accuracy $\varepsilon>0$: $\left|\tilde{\overline{V}}-\overline{V}\right|\leqslant\varepsilon$. Alas, for any ε, such computations are also NP-hard:

THEOREM 2. *For every $\varepsilon>0$, the problem of computing $\overline{V}$ with accuracy ε is NP-hard.*

It is worth mentioning the $\overline{V}$ can be computed exactly in exponential time $O(2^n)$:

THEOREM 3. *There exists an algorithm that computes $\overline{V}$ in exponential time.*

For computing $\underline{V}$, there is a feasible algorithm: specifically, our algorithm is *quadratic time*, i.e., it requires $O(n^2)$ computational steps (arithmetic operations or comparisons) for n interval data points $\mathbf{x}_i=[\underline{x}_i,\overline{x}_i]$.

The algorithm $\underline{\mathcal{A}}$ is as follows:

- First, we sort all $2n$ values $\underline{x}_i,\overline{x}_i$ into a sequence $x_{(1)}\leqslant x_{(2)}\leqslant\cdots\leqslant x_{(2n)}$.
- Second, we compute $\underline{E}$ and $\overline{E}$ and select all "small intervals" $[x_{(k)},x_{(k+1)}]$ that intersect with $[\underline{E},\overline{E}]$.

- For each of the selected small intervals $[x_{(k)}, x_{(k+1)}]$, we compute the ratio $r_k = S_k/N_k$, where
$$S_k \stackrel{\text{def}}{=} \sum_{i:\underline{x}_i \geqslant x_{(k+1)}} \underline{x}_i + \sum_{j:\overline{x}_j \leqslant x_{(k)}} \overline{x}_j$$
and N_k is the total number of such i's and j's If $r_k \in [x_{(k)}, x_{(k+1)}]$, then we compute
$$V_k \stackrel{\text{def}}{=} \frac{1}{n} \cdot \left(\sum_{i:\underline{x}_i \geqslant x_{(k+1)}} (\underline{x}_i - r_k)^2 + \sum_{j:\overline{x}_j \leqslant x_{(k)}} (\overline{x}_j - r_k)^2 \right).$$
If $N_k = 0$, we take $V_k \stackrel{\text{def}}{=} 0$.
- Finally, we return the smallest of the values V_k as $\underline{V}$.

THEOREM 4. *The algorithm $\underline{\mathcal{A}}$ always compute $\underline{V}$ is quadratic time.*

NP-hardness of computing $\overline{V}$ means, crudely speaking, that there are no general ways for solving all particular cases of this problem (i.e., computing $\overline{V}$) in reasonable time.

However, we show that there are algorithms for computing $\overline{V}$ for many reasonable situations. Namely, we propose an efficient algorithm that computes $\overline{V}$ for the case when all the interval midpoints ("measured values") $\tilde{x}_i = (\underline{x}_i + \overline{x}_i)/2$ are definitely different from each other, in the sense that the "narrowed" intervals $[\tilde{x}_i - \Delta_i/n, \tilde{x}_i + \Delta_i/n]$ — where $\Delta_i = (\underline{x}_i - \overline{x}_i)/2$ is the interval's half-width — do not intersect with each other.

The algorithm $\overline{\mathcal{A}}$ is as follows:

- First, we sort all $2n$ endpoints of the narrowed intervals $\tilde{x}_i - \Delta_i/n$ and $\tilde{x}_i + \Delta_i/n$ into the sequence $x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(2n)}$. This enables us to divide the real line into $2n+1$ segments ("small intervals") $[x_{(k)}, x_{(k+1)}]$, where we denoted $x_{(0)} \stackrel{\text{def}}{=} -\infty$ and $x_{(2n+1)} \stackrel{\text{def}}{=} +\infty$.
- Second, we compute $\underline{E}$ and $\overline{E}$ and pick all "small intervals" $[x_{(k)}, x_{(k+1)}]$ that intersect with $[\underline{E}, \overline{E}]$.
- For each of remaining small intervals $[x_{(k)}, x_{(k+1)}]$, for each i from 1 to n, we pick the following value of x_i:
  - if $x_{(k+1)} < \tilde{x}_i - \Delta_i/n$, then we pick $x_i = \overline{x}_i$;
  - if $x_{(k)} > \tilde{x}_i + \Delta_i/n$, then we pick $x_i = \underline{x}_i$;
  - for all other i, we consider both possible values $x_i = \overline{x}_i$ and $x_i = \underline{x}_i$.

  As a result, we get one or several sequences of x_i. For each of these sequences, we check whether the average E of the selected values $x_1, \ldots, x_n$ is indeed within this small interval, and if it is, compute the variance by using the formula (1).
- Finally, we return the largest of the computed variances as $\overline{V}$.

THEOREM 5. *The algorithm $\overline{\mathcal{A}}$ computes $\overline{V}$ is quadratic time for all the cases in which the "narrowed" intervals do not intersect with each other.*

The algorithm also works when, for some fixed k, no more than k "narrowed" intervals can have a common point:

THEOREM 6. *For every positive integer k, the algorithm $\overline{\mathcal{A}}$ computes $\overline{V}$ is quadratic time for all the cases in which no more than k "narrowed" intervals can have a common point.*

3. Important example: Interval computations related to privacy in statistical databases

2.1. NEED FOR PRIVACY

Privacy is an important issue in the statistical analysis of human-related data. For example, to check whether in a certain geographic area, there is a gender-based discrimination, we can use the census data to check, e.g., whether for all people from this area who have the same level of education, there is a correlation between salary and gender. One can think of numerous possible questions of this type related to different sociological, political, medical, economic, and other questions. From this viewpoint, it is desirable to give researches *ability to perform* whatever *statistical analysis* of this data that is reasonable for their specific research.

On the other hand, we do not want to give them direct access to the raw census data, because a large part of the census data is *confidential*. For example, for most people (those who work in the private sector) salary information is confidential. Suppose that a corporation is deciding where to build a new plant and has not yet decided between two possible areas. This corporation would benefit from knowing the average salary of people of needed education level in these two areas, because this information would help them estimate how much it will cost to bring local people on board. However, since salary information is confidential, the company should not be able to know the exact salaries of different potential workers.

The need for privacy is also extremely important for *medical* experiments, where we should be able to make statistical conclusions about, e.g., the efficiency of a new medicine without disclosing any potentially embarrassing details from the individual medical records.

Such databases in which the outside users have cannot access individual records but can solicit statistical information are often called *statistical databases*.

3.2. PRIVACY LEADS TO INTERVALS

A natural way to fully describe a single real-valued random variable η is to provide the values of its cumulative density function (CDF)

$$F(x) = \text{Prob}(\eta \leqslant x)$$

for all possible real numbers x. Once we know $F(x)$, we can determine the values of all possible statistical characteristics of this random variable — e.g., its first moment, second moment, variance, etc. This, it is natural to allow the users to solicit the values of $F(x)$ for different x; from this information, the users will be able to reconstruct all other statistical characteristics.

For discrete data $x_1, \ldots, x_n$, the corresponding sample distribution — in which each value x_i occurs with probability $1/n$ — is described by the CDF $F(x)$ for which

$$F(x) = (1/n) \cdot \#\{i : x_i \leqslant x\}.$$

To get the full information about the data, we should allow the user to ask for the values $F(x)$ for all possible real numbers x. However, once we know the values $F(x)$ for all x, we can determine all the values x_i. Thus, if we want to keep privacy, we must only allow the users to know $F(x)$ for some fixed values $x^{(1)} \leqslant \cdots \leqslant x^{(m)}$. This way, instead of the actual values x_i, all we know is an *interval* $[x^{(k)}, x^{(k+1)}]$ that contains x_i. Intervals corresponding to different values are *almost disjoint*, i.e., either disjoint (intersect in at most one point) or identical. How can we compute statistical characteristics based on this information?

THEOREM 7. *There exists a quadratic-time algorithm that computes the exact range* $\mathbf{V}$ *of variance* V *for the case when intervals* $\mathbf{x}_i$ *of possible values of* x_i *are pairwise almost disjoint.*

Proof. Since there exists an algorithm that computes $\underline{V}$ in feasible time, it is sufficient to produce a feasible algorithm for computing $\overline{V}$.

According to the proof of Theorems 4.1 and 4.2 from [1], the values $x_i \in \mathbf{x}_i$ that lead to the largest possible value of V satisfy the following property:

- if $E \leqslant \underline{x}_i$, then $x_i = \overline{x}_i$;
- if $E \leqslant \overline{x}_i$, then $x_i = \underline{x}_i$;
- if $E \in (\underline{x}_i, \overline{x}_i)$, then $x_i = \underline{x}_i$ or $x_i = \overline{x}_i$.

In order to use this property to compute $\overline{V}$, we test all possible locations of E in relation to the intervals $\mathbf{x}_i$: $E = \underline{x}_i$, $E = \overline{x}_i$ and $E \in (\underline{x}_i, \overline{x}_i)$ for different $i = 1, 2, \ldots, n$.

Let us first consider the cases when $E = \underline{x}_i$ (the case when $E = \overline{x}_i$ is treated similarly). In these case, since the intervals $\mathbf{x}_i$ are almost disjoint, the above property uniquely determines the values x_i; thus, we can compute E, check whether it indeed satisfies the corresponding condition, and if yes, compute the corresponding value V.

Let us now consider the cases when $E \in (\underline{x}_i, \overline{x}_i)$. Let k denote the number of different intervals of such type, and let x_j, $j = 1, \ldots, k$ denote the number of intervals $\mathbf{x}_i$ that coincide with j-th interval. Then, $n = n_1 + \cdots + n_k$. For each of these k intervals $\mathbf{x}_j$, the values of x_i are uniquely determined when $\overline{x}_j \leqslant \underline{x}_i$ or $\overline{x}_i \leqslant \underline{x}_j$; for the remaining n_j values x_i, we have $x_i = \underline{x}_i$ or $x_i = \overline{x}_i$. Modulo transposition, the resulting set of values $\{x_1, \ldots, x_n\}$ is uniquely determined by how many of these n_j x_i's are equal to $\overline{x}_i$. The number of such x_i's can be $0, 1, 2, \ldots, n_j + 1$. Thus, the total number of such combinations is equal to $n_j + 1$. Overall, for all j from 1 to k, we have $\sum_{j=1}^{k} (n_j + 1) = \sum_{j=1}^{k} n_j + k = n + k \leqslant 2n$ resulting sets $\{x_1, \ldots, x_n\}$. For each of these sets, we compute E, check that the resulting E is indeed inside the corresponding interval $\mathbf{x}_i$, and if it is, we compute V.

Thus, we have $\leqslant 2n + n = 3n$ cases, for each of which we need $O(n)$ computations to compute V. The largest of these V is the desired $\overline{V}$, and we compute it in time $\leqslant 3n \cdot O(n) = O(n^2)$. The proposition is proven. □

Comment. Similar algorithms can be providing for computing the exact range of covariance between two interval-valued data sequences; in general, the problem of computing the range for covariance is NP-hard [1].

4. Second step beyond intervals: extension of interval arithmetic to situations with partial information about probabilities

4.1. PRACTICAL PROBLEM

In some practical situations, in addition to the lower and upper bounds on each random variable x_i, we know the bounds $\mathbf{E}_i = [\underline{E}_i, \overline{E}_i]$ on its mean E_i

Indeed, in measurement practice (see e.g. [14]), the overall measurement error Δx is usually represented as a sum of two components:

- a *systematic* error component $\Delta_s x$ which is defined as the expected value $E[\Delta x]$, and
- a *random* error component $\Delta_r x$ which is defined as the difference between the overall measurement error and the systematic error component $\Delta_r x \stackrel{\text{def}}{=} \Delta x - \Delta_s x$.

In addition to the bound Δ on the overall measurement error, the manufacturers of the measuring instrument often provide an upper bound Δ_s on the systematic error component: $|\Delta_s x| \leqslant \Delta_s$.

The additional information is provided because, with this additional information, we not only get a bound on the accuracy of a single measurement, but we also get an idea of what accuracy we can attain if we use repeated measurements to increase the measurement accuracy. Indeed, the very idea that repeated measurements can improve the measurement accuracy is natural: we measure the same quantity by using the same measurement instrument several (N) times, and then

take, e.g., an arithmetic average $\overline{x}=(\tilde{x}^{(1)}+\cdots+\tilde{x}^{(N)})/N$ of the corresponding measurement results $\tilde{x}^{(1)}=x+\Delta x^{(1)},\ldots,\tilde{x}^{(N)}=x+\Delta x^{(N)}$.

- If systematic error is the only error component, then all the measurements lead to exactly the same value $\tilde{x}^{(1)}=\ldots\tilde{x}^{(N)}$, and averaging does not change the value — hence does not improve the accuracy.
- On the other hand, if we know that the systematic error component is 0, i.e., $E[\Delta x]=0$ and $E[\tilde{x}]=x$, then, as $N\to\infty$, the arithmetic average tends to the actual value x. In this case, by repeating the measurements sufficiently many times, we can determine the actual value of x with an arbitrary given accuracy.

In general, by repeating measurements sufficiently many times, we can arbitrarily decrease the random error component and thus attain accuracy as close to Δ_s as we want.

When this additional information is given, then, after we performed a measurement and got a measurement result $\tilde{x}$, then not only we get the information that the actual value x of the measured quantity belongs to the interval $\mathbf{x}=[\tilde{x}-\Delta,\tilde{x}+\Delta]$, but we can also conclude that the expected value of $x=\tilde{x}-\Delta x$ (which is equal to $E[x]=\tilde{x}-E[\Delta x]=\tilde{x}-\Delta_s x$) belongs to the interval $\mathbf{E}=[\tilde{x}-\Delta_s,\tilde{x}+\Delta_s]$.

If we have this information for every x_i, then, in addition to the interval $\mathbf{y}$ of possible value of y, we would also like to know the interval of possible values of $E[y]$. This additional interval will hopefully provide us with the information on how repeated measurements can improve the accuracy of this indirect measurement. This, we arrive at the following problem.

4.2. RESULTING OPTIMIZATION PROBLEM

In more optimization terms, we want to solve the following problem: given an algorithm computing a function $f(x_1,\ldots,x_n$ from R^n to R; and values $\underline{x}_1,\overline{x}_1,\ldots,\underline{x}_n,\overline{x}_n,\underline{E}_1,\overline{E}_1,\ldots,\underline{E}_n,\overline{E}_n$, we want to find

$$\underline{E}\stackrel{\text{def}}{=}\min\{E[f(x_1,\ldots,x_n)]\,|\,\text{all distributions of }(x_1,\ldots,x_n)\text{ for which}\\ x_1\in[\underline{x}_1,\overline{x}_1],\ldots,x_n\in[\underline{x}_n,\overline{x}_n],\ E[x_1]\in[\underline{E}_1,\overline{E}_1],\ldots E[x_n]\in[\underline{E}_n,\overline{E}_n]\};$$

and $\overline{E}$ which is the maximum of $E[f(x_1,\ldots,x_n)]$ for all such distributions.

In addition to considering all possible distributions, we can also consider the case when all the variables x_i are independent.

4.3. ANALOG OF STRAIGHTFORWARD INTERVAL COMPUTATIONS

The main idea behind straightforward interval computations can be applied here as well. Namely, first, we find out how to solve this problem for the case when $n=2$ and $f(x_1,x_2)$ is one of the standard arithmetic operations. Then, once we

have an arbitrary algorithm $f(x_1,\ldots,x_n)$, we parse it and replace each elementary operation on real numbers with the corresponding operation on quadruples $(\underline{x},\underline{E},\overline{E},\overline{x})$.

To implement this idea, we must therefore know how to solve the above problem for elementary operations.

4.4. SOLUTION

For *addition*, the answer is simple. Since $E[x_1+x_2]=E[x_1]+E[x_2]$, if $y=x_1+x_2$, there is only one possible value for $E=E[y]$: the value $E=E_1+E_2$. This value does not depend on whether we have correlation or not, and whether we have any information about the correlation. Thus, $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$.

Similarly, the answer is simple for *substraction:* if $y=x_1-x_2$, there is only one possible value for $E=E[y]$: the value $E=E_1-E_2$. Thus, $\mathbf{E}=\mathbf{E}_1-\mathbf{E}_2$.

For *multiplication*, if the variables x_1 and x_2 are independent, then $E[x_1\cdot x_2]=E[x_1]\cdot E[x_2]$. Hence, if $y=x_1\cdot x_2$ and x_1 and x_2 are independent, there is only one possible value for $E=E[y]$: the value $E=E_1\cdot E_2$; hence $\mathbf{E}=\mathbf{E}_1\cdot\mathbf{E}_2$.

The first non-trivial case is the case of multiplication in the presence of possible correlation. When we know the exact values of E_1 and E_2, the solution to the above problem is as follows:

THEOREM 8. *For multiplication $y=x_1\cdot x_2$, when we have no information about the correlation.*

$$\begin{aligned}\underline{E} = {} & \max(p_1+p_2-1,0)\cdot\overline{x}_1\cdot\overline{x}_2+\min(p_1,1-p_2)\cdot\overline{x}_1\cdot\underline{x}_2\\ & +\min(1-p_1,p_2)\cdot\underline{x}_1\cdot\overline{x}_2+\max(1-p_1-p_2,0)\cdot\underline{x}_1\cdot\underline{x}_2;\end{aligned}$$

and

$$\begin{aligned}\overline{E} = {} & \min(p_1,p_2)\cdot\overline{x}_1\cdot\overline{x}_2+\max(p_1-p_2,0)\cdot\overline{x}_1\cdot\underline{x}_2\\ & +\max(p_2-p_1,0)\cdot\underline{x}_1\cdot\overline{x}_2+\min(1-p_1,1-p_2)\cdot\underline{x}_1\cdot\underline{x}_2.\end{aligned}$$

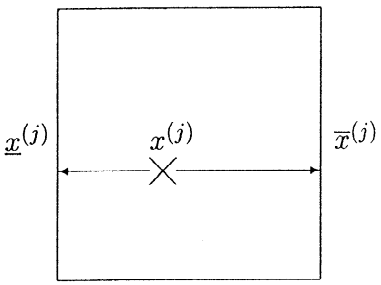
where $p_i\stackrel{\text{def}}{=}(E_i-\underline{x}_i)/(\overline{x}_i-\underline{x}_i)$.

Proof. Let us show that a general distribution with $E[x_i]=E_i$ can be simplified without changing the values $E[x_i]$ and $E[x_1\cdot x_2]$. Thus, to describe possible values of $E[x_1\cdot x_2]$, we do not need to consider all possible distributions, it is sufficient to consider only the simplified ones.

We will describe the simplification for discrete distribution that concentrate on finitely many points $x^{(j)}=(x_1^{(j)},x_2^{(j)}),1\leqslant j\leqslant N$. An arbitrary probability distribution can be approximated by such distributions, so we do not lose anything by this restriction.

So, we have a probability distribution in which the point $x^{(1)}$ appears with the probability $p^{(1)}$, the point $x^{(2)}$ appears with the probability $p^{(2)}$, etc. Let us modify this distribution as follows: pick a point $x^{(j)}=(x_1^{(j)},x_2^{(j)})$ that occurs with probability $p^{(j)}$, and replace it with two points: $\overline{x}^{(j)}=(\overline{x}_1,x_2^{(j)})$ with probability $p^{(j)}\cdot\overline{p}^{(j)}$

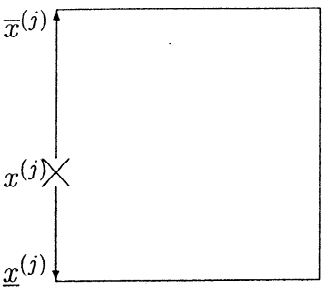
and $\underline{x}^{(j)} = (\underline{x}_1, x_2^{(j)})$ with probability $p^{(j)} \cdot \underline{p}^{(j)}$, where $\overline{p}^{(j)} \stackrel{\text{def}}{=} (x_1^{(j)} - x_1)/(\overline{x}_1 - \underline{x}_1)$ and $\underline{p}^{(j)} \stackrel{\text{def}}{=} 1 - \overline{p}^{(j)}$:



Here, the values $\overline{p}^{(j)}$ and $\underline{p}^{(j)} = 1 - \overline{p}^{(j)}$ are chosen in such a way that $\overline{p}^{(j)} \cdot \overline{x}_1 + \underline{p}^{(j)} \cdot \underline{x}_1 = x_1^{(j)}$. Due to this choice, $p^{(j)} \cdot \overline{p}^{(j)} \cdot \overline{x}_1 + p^{(j)} \cdot \underline{p}^{(j)} \cdot \underline{x}_1 = p^{(j)} \cdot x_1^{(j)}$, hence for the new distribution, the mathematical expectation $E[x_1]$ is the same as for the old one. Similarly, we can prove that the values $E[x_2]$ and $E[x_1 \cdot x_2]$ do not change.

We started with a general discrete distribution with N points for each of which $p_1^{(j)}$ could be inside the interval $\mathbf{x}_1$, and we have a new distribution for which $\leqslant N-1$ points have the value x_1 inside this interval. We can perform a similar replacement for all N points and get a distribution with the same values of $E[x_1]$, $E[x_2]$, and $E[x_1 \cdot x_2]$ as the original one but for which, for every point, x_1 is equal either to $\underline{x}_1$, or to $\overline{x}_1$.

For the new distribution, we can perform a similar transformation relative to x_1 and end up — without changing the values x_1 — with the distribution for which always either $x_2 = \underline{x}_1$ or $x_2 = \overline{x}_2$:



Thus, instead of considering all possible distributions, it is sufficient to consider only distributions for which $x_1 \in \{\underline{x}_1, \overline{x}_1\}$ and $x_2 \in \{\underline{x}_2, \overline{x}_2\}$. In other words, it is sufficient to consider only distributions which are located in the four corner points $(\underline{x}_1, \underline{x}_2)$, $(\underline{x}_1, \overline{x}_2)$, $(\overline{x}_1, \underline{x}_2)$ and $(\overline{x}_1, \overline{x}_2)$ of the box $\mathbf{x}_1 \times \mathbf{x}_2$.

Such distribution can be characterized by the probabilities of these four points. These four probabilities must satisfy 3 conditions: that their sum is 1, that $E[x_1]$ is E_1, and that $E[x_2]=E_2$. Thus, we only have one parameter left; optimizing with respect to this parameter, we get the desired formulas for $\underline{E}$ and $\overline{E}$. The theorem is proven. □

When we only know the intervals $\mathbf{E}_i$ of possible values of E_i, instead of the values p_i, we have the corresponding intervals $\mathbf{p}_i=(\mathbf{E}_i-\underline{x}_i)/(\overline{E}_i-\underline{x}_i)$. In terms of these intervals, we get the following results

THEOREM 9. *For multiplication under no information about dependence, to find $\underline{E}$, it is sufficient to consider the following combinations of p_1 and p_2:*

- $p_1=\underline{p}_1$ *and* $p_2=\underline{p}_2$; $p_1=\underline{p}_1$ *and* $p_2=\overline{p}_2$; $p_1=\overline{p}_1$ *and* $p_2=\underline{p}_2$; $p_1=\overline{p}_1$ *and* $p_2=\overline{p}_2$
- $p_1=\max(\underline{p}_1, 1-\overline{p}_2)$ *and* $p_2=1-p_1$ (*if* $1\in\mathbf{p}_1+\mathbf{p}_2$); *and*
- $p_1=\min(\overline{p}_1, 1-\underline{p}_2)$ *and* $p_2=1-p_1$ (*if* $1\in\mathbf{p}_1+\mathbf{p}_2$).

The smallest value of $\underline{E}$ for all these cases is the desired lower bound $\underline{E}$.

THEOREM 10. *For multiplication under no information about dependence, to find $\overline{E}$, it is sufficient to consider the following combinations of p_1 and p_2:*

- $p_1=\underline{p}_1$ *and* $p_2=\underline{p}_2$; $p_1=\underline{p}_1$ *and* $p_2=\overline{p}_2$; $p_1=\overline{p}_1$ *and* $p_2=\underline{p}_2$; $p_1=\overline{p}_1$ *and* $p_2=\overline{p}_2$;
- $p_1=p_2=\max(\underline{p}_1, \underline{p}_2)$ (*if* $\mathbf{p}_1\cap\mathbf{p}_2\neq\emptyset$); *and*
- $p_1=p_2=\min(\overline{p}_1, \overline{p}_2)$ (*if* $\mathbf{p}_1\cap\mathbf{p}_2\neq\emptyset$).

The largest value of $\overline{E}$ for all these cases is the desired upper bound $\overline{E}$.

Proof. We will prove Theorem 10; the proof of Theorem 9 is similar. The formula for $\overline{E}$ given in Theorem 8 can be simplified if we consider two cases: $p_1\leqslant p_2$ and $p_1\geqslant p_2$. To find the largest possible value $\overline{E}$ of E, it is sufficient to consider the largest possible values for each of these cases, and then take the largest of the resulting two numbers.

In each case, for a fixed p_2, the formula is linear in p_1. To find the maximum of a linear function on an interval, it is sufficient to consider this interval's endpoints. Thus, the maximum in p_1 is attained when either p_1 attains its smallest possible value $\underline{p}_1$, or when p_1 attains the largest possible value within this case; depending on p_2, this value is either $p_1=\overline{p}_1$ or $p_1=p_2$.

Thus, to find the maximum for each cases, it is sufficient to consider only the following cases: $p_1=\underline{p}_1$, $p_1=\overline{p}_1$, and $p_1=p_2$. Similarly, it is sufficient to consider only the following cases for p_2: $p_2=\underline{p}_2$, $p_2=\overline{p}_2$ and $p_1=p_2$.

When $p_1=p_2$, the probability $p_1=p_2$ can take all possible values from the intersection $\mathbf{p}_1\cap\mathbf{p}_2$. The formula for $\overline{E}$ is linear in p_1, so to find its maximum, it is sufficient to consider the endpoints of the interval $\mathbf{p}_1\cap\mathbf{p}_2$, i.e., the values $p_1=p_2=\max(\underline{p}_1, \underline{p}_2)$ and $p_1=p_2=\min(\overline{p}_1, \overline{p}_2)$. The theorem is proven. □

For the *inverse* $y=1/x$, bounds for E can be deduced from convexity [15]: $\mathbf{E}=[1/E_1, p_1/\underline{x}_1+(1-p_1)/x_1]$.

For min and independent x_i, we have $\overline{E}=\min(E_1,E_2)$ and

$$\begin{aligned}\underline{E} = {} & p_1\cdot p_2\cdot\min(\overline{x}_1,\overline{x}_2)+p_1\cdot(1-p_2)\cdot\min(\overline{x}_1,\underline{x}_2)\\ & +(1-p_1)\cdot p_2\cdot\min((\underline{x}_1,\overline{x}_2)+(1-p_1)\cdot(1-p_2)\cdot\min(\underline{x}_1,\underline{x}_2).\end{aligned}$$

For max and independent x_i, we have $\underline{E}=\max(E_1,E_2)$ and

$$\begin{aligned}\overline{E} = {} & p_1\cdot p_2\cdot\max(\overline{x}_1,\overline{x}_2)+p_1\cdot(1-p_2)\cdot\max(\overline{x}_1,\underline{x}_2)\\ & +(1-p_1)\cdot p_2\cdot\max((\underline{x}_1,\overline{x}_2)+(1-p_1)\cdot(1-p_2)\cdot\max(\underline{x}_1,\underline{x}_2).\end{aligned}$$

For min in the general case, $\overline{E}=\min(E_1,E_2)$,

$$\begin{aligned}\underline{E} = {} & \max(p_1+p_2-1,0)\cdot\min(\overline{x}_1,\overline{x}_2)+\min(p_1,1-p_2)\cdot\min(\overline{x}_1,\underline{x}_2)\\ & +\min(1-p_1,p_2)\cdot\min(\underline{x}_1,\overline{x}_2)+\max(1-p_1-p_2,0)\cdot\min(\underline{x}_1,\underline{x}_2).\end{aligned}$$

For max in the general case, $\underline{E}=\max(E_1,E_2)$ and

$$\begin{aligned}\overline{E} = {} & \min(p_1,p_2)\cdot\max(\overline{x}_1,\overline{x}_2)+\max(p_1-p_2,0)\cdot\max(\overline{x}_1,\underline{x}_2)\\ & +\max(p_2-p_1,0)\cdot\max(\underline{x}_1,\overline{x}_2)+\min(1-p_1,1-p_2)\cdot\max(\underline{x}_1,\underline{x}_2).\end{aligned}$$

Similar formulas can be produced for the cases when there is a strong correlation between x_i: namely, when x_1 is (non-strictly) increasing or decreasing in x_2.

4.5. FROM ELEMENTARY ARITHMETIC OPERATIONS TO GENERAL ALGORITHMS

When we have a complex algorithm f, then a step-by-step approach leads to excess width. How can we find the actual range of $E=E[y]$?

At first glance, the exact formulation of this problem requires that we use finitely many variables, because we must describe all possible probability distributions on the box $\mathbf{x}_1\times\cdots\times\mathbf{x}_n$ (or, in the independent case, all possible tuples consisting of distributions on all n intervals $\mathbf{x}_1,\ldots,\mathbf{x}_n$). It turns out, however, that we can reformulate these problems in equivalent forms that require only finitely many variables:

THEOREM 11. [9] *For a general continuous function $f(x_1,\ldots,x_n)$, $\underline{E}$ is a solution to the following optimization problem*: $\sum_{j=0}^{n} p^{(j)}\cdot f(x_1^{(j)},\ldots,x_n^{(j)})\to\min$ *under the conditions*

$$\sum_{k=0}^{n} p^{(k)}=1;\ p^{(j)}\geqslant 0;\ \underline{x}_i\leqslant x_i^{(j)}\leqslant\overline{x}_i;\ \underline{E}_i\leqslant\sum_{j=0}^{n} p^{(j)}\cdot x_i^{(j)}\leqslant\overline{E}_i\ (\text{for all } i,j),$$

and $\overline{E}$ is a solution to $\sum_{j=0}^{n} p^{(j)}\cdot f(x_1^{(j)},\ldots,x_n^{(j)})\to\max$ under the same constraints.

Proof. In terms of the unknown probabilities $p^{(j)}$, we are minimizing a linear function under linear constraints (equalities and inequalities). Geometrically, the set of all points that satisfy several linear constraints is a polytope. It is well known that to find the minimum of a linear function on a polytope, it is sufficient to consider its vertices (this idea is behind linear programming). In algebraic terms, a vertex can be characterized by the fact that for N variables, N of the original constraints are equalities. Thus, in our case, all but n probabilities $p^{(j)}$ must be equal to 0. The theorem is proven. □

5. Open problems

So far, we have provided explicit formulas for the elementary arithmetic operations $f(x_1,\ldots,x_n)$ for the case when we know the first order moments. What if, in addition to that, we have some information about second order (and/or higher order) moments of x_i? What will we be then able to conclude about the moments of y? Partial answers to this question are given in [9,15,17]; it is desirable to find a general answer.

Similarly to Theorem 11, we can reduce the corresponding problems to the constraint optimization problems with finitely many variables. For example, when in addition to intervals $\mathbf{E}_i$ that contain the first moments $E[x_i]$, we know the intervals $\mathbf{E}_{ik}$ that contain the second moments $E[x_i\cdot x_k]$, then the corresponding bounds $\underline{E}$ and $\overline{E}$ on $E[y]$ can be computed by solving the problems $\sum_{j=0}^{N} p^{(j)}\cdot f(x_1^{(j)},\ldots,x_n^{(j)})\to \min(\max)$ under the conditions

$$\sum_{j=0}^{N} p^{(j)}=1;\quad p^{(j)}\geqslant 0;\quad \underline{x}_i\leqslant x_i^{(j)}\leqslant \overline{x}_i;\quad \underline{E}_i\leqslant\sum_{j=0}^{n} p^{(j)}\cdot x_i^{(j)}\leqslant\overline{E}_i;$$

$$\underline{E}_{ik}\leqslant\sum_{j=0}^{n} p^{(j)}\cdot x_i^{(j)}\cdot x_k^{(j)}\leqslant\overline{E}_{ik},$$

where $N=n(n+1)/2$.

It is desirable to find explicit analytical expressions for these bounds, at least for the case $n=2$ and $f(x_1,\ldots,x_n)$ is an elementary arithmetic operation.

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